

Monotonicity and Unbiasedness Properties of  
ANOVA and MANOVA Tests.<sup>(1)</sup>

Somesh Das Gupta\*  
University of Minnesota

Technical Report No. 318

- (1) To be published in "Handbook of Statistics, Vol. 1: Analysis of Variance" by North-Holland Publishing Co.

\*Partially supported by a grant from the Mathematics Division, U.S. Army Research Office, Durham, N.C. Grant no. DAAG-29-76-G-0038.

## 1. Introduction

The multivariate analysis of variance problem for the normal case may be posed as follows: Let  $X: p \times n$  be a random matrix such that its column vectors are independently distributed as  $n_p(\cdot, \Sigma)$ , where  $\Sigma$  is an unknown positive-definite matrix; moreover,

$$(1.1) \quad \mathcal{E}(X') = A\Theta,$$

where  $A: n \times m$  is a known matrix of rank  $r$  and  $\Theta: m \times p$  is a matrix of unknown parameters. The problem is to test  $H_0: G'\Theta = 0$  against  $H_1: G'\Theta \neq 0$ , where  $G'$  is a known  $s \times m$  matrix of rank  $s$  such that  $G = A'B$  for some  $B: n \times s$ . This problem can easily be reduced to the following canonical form: Let  $Y_1, \dots, Y_n$  be  $n$  independently distributed  $p \times 1$  random vectors such that

$Y_\alpha \sim n_p(\mu_\alpha, \Sigma)$ , where  $\mu_{r+1} = \dots = \mu_n = 0$ , and  $\Sigma$  along with  $\mu_1, \dots, \mu_r$  are unknown,  $\Sigma$  being positive-definite. The problem is to test

$$(1.2) \quad H_0: \mu_1 = \dots = \mu_s = 0$$

against  $H_1: \text{"not } H_0 \text{"}$ , where  $s \leq r$ . In this set-up  $S_0 \equiv \sum_{\alpha=1}^s Y_\alpha Y_\alpha'$  and

$S_e = \sum_{\alpha=r+1}^n Y_\alpha Y_\alpha'$  are called the sums of products (s.p.) matrices due to

the hypothesis  $H_0$  and error, respectively; the corresponding degrees of freedom are  $s$  and  $n_e \equiv n-r$ .

The following tests (represented by their acceptance regions) are most-often considered in the literature:

(a) Likelihood-ratio test:

$$(1.3) \quad \det(S_e) / \det(S_e + S_0) \geq k_1 \quad (0 < k_1 < 1)$$

(b) Roy's maximum-root test:

$$(1.4) \quad \max[\text{characteristic root of } S_0 S_e^{-1}] \leq k_2 \quad (0 < k_2)$$

(c) Lawley-Hotelling's trace test:

$$(1.5) \quad \text{tr}(S_0 S_e^{-1}) \leq k_3 \quad (0 < k_3)$$

(d) Pillai's trace test:

$$(1.6) \quad \text{tr}[S_0(S_0 + S_e)^{-1}] \leq k_4 \quad (0 < k_4 < \min(p, s)).$$

Note that the first three tests are defined only when  $n_e \geq p$  in which case  $S_e$  is non-singular with probability 1. The last test is defined when  $n_e + s > p$ . All these four tests are members of a class of invariant tests which is defined as follows.

Let

$$(1.7) \quad \begin{aligned} Y_{(1)} &= (Y_1, \dots, Y_s), \quad Y_{(2)} = (Y_{s+1}, \dots, Y_r), \\ Y_{(3)} &= (Y_{r+1}, \dots, Y_n). \end{aligned}$$

A set of sufficient statistics is given by

$$(1.8) \quad (Y_{(1)}, Y_{(2)}, S_t \equiv Y_{(1)} Y_{(1)}' + Y_{(3)} Y_{(3)}').$$

$S_t$  is positive-definite when  $n_e + s \geq p$ . Consider the following transformation:

$$(1.9) \quad (L, A, B) (Y_{(1)}, Y_{(2)}, S_t) = (A Y_{(1)}^L, A Y_{(2)}^{+B}, A S_t A'),$$

where

$L \in \mathcal{O}_s$  = the class of all  $s \times s$  orthogonal matrices,

$A \in \mathcal{L}_p$  = the class of all  $p \times p$  nonsingular matrices,

$B \in \mathcal{M}_{p, r-s}$  = the class of all  $p \times (r-s)$  matrices.

This transformation keeps the above model and the testing problem invariant. The composition of two such transformations is given by

$$(1.10) \quad (L_1, A_1, B_1)(L_2, A_2, B_2) = (L_2 L_1, A_1 A_2, A_1 B_2 + B_1) .$$

Then the collection  $G$  of  $(L, A, B)$  with the above binary operation is a group of transformations acting on  $M_{p,s} \times M_{p,r-s} \times S_p^+$ , where

$S_p^+$  = the collection of all  $p \times p$  positive-definite matrices.

Let  $\bar{\Phi}_G$  be the class of all non-randomized tests invariant under  $G$ .

Lemma. When  $n_e + s > p$ , a set of maximal invariants under  $G$  in the space of sufficient statistics  $(Y_{(1)}, Y_{(2)}, S_t)$  is given by the ordered non-zero characteristic roots of  $S_0 S_t^{-1}$ , denoted by  $d_1 < \dots < d_\ell$  where  $\ell = \min(s, p)$ . When  $n_e + s \leq p$  there is no non-trivial invariant test.

Suppose  $n_e \geq p$  and let  $c_1 \geq \dots \geq c_\ell$  be the ordered non-zero characteristic roots of  $S_0 S_e^{-1}$ . Then  $d_i = c_i / (1 + c_i)$ .

Next we shall consider two important special cases.

(I)  $s = 1, n_e \geq p$ . The acceptance regions of all the above four tests reduce to

$$(1.11) \quad c_1 = Y_1' (Y_{(3)} Y_{(3)}')^{-1} Y_1 \leq k .$$

This is the UMP invariant test for its size.

(II)  $p = 1, n_e \geq 1$ . The acceptance regions of all the above four tests reduce to

$$(1.12) \quad \frac{\sum_{\alpha=1}^s Y_\alpha^2}{\sum_{\alpha=r+1}^n Y_\alpha^2} \leq k .$$

This is also the UMP invariant test for its size.

Except for these two special cases UMP invariant test does not exist. All the above four tests are known to be admissible. Instead of comparing the power functions of different tests we shall be concerned in this paper with the behavior of the power function of a given test

with respect to the non-centrality parameters involved; in particular we shall study whether the unbiasedness property is satisfied by a given test.

Let  $\tau_1^2, \dots, \tau_\ell^2$  be the possible non-zero characteristic roots of  $\Sigma^{-1}MM'$ , where  $M = (\mu_1, \dots, \mu_s)$ . Then the power function of any test in  $\mathfrak{G}$  involves  $M$  and  $\Sigma$  only through  $\tau_1^2, \dots, \tau_\ell^2$ . We shall study conditions under which the power function of an invariant test increases monotonically in each  $\tau_i^2$ . Under some additional conditions we shall get a more refined property of this monotonicity.

## 2. Monotonicity of the power functions of the UMP invariant tests in the two special cases.

The monotonicity property in the above two special cases can be easily proved using the following elementary result.

Theorem 2.1: Let  $Z$  be a random variable distributed as  $N(0,1)$ . Then

$$(2.1) \quad \pi(\tau) \equiv P\{|Z + \tau| \leq k\}$$

for  $k > 0$  is a symmetric function of  $\tau$  and decreases monotonically as  $\tau^2$  increases.

The theorem is proved easily by studying the first derivative of  $\pi$  with respect to  $\tau$ . Later we shall show that this result also holds when the density of  $Z$  is symmetric about the origin and unimodal (with the mode at the origin). It will also be extended to the multivariate case.

Corollary 2.1. Let  $Z_1$  and  $Z_2$  be independently distributed according to the non-central chi-square  $\chi_{n_1}^2(\tau^2)$  and the central chi-square  $\chi_{n_2}^2$  distributions, respectively;  $n_1$  and  $n_2$  are positive integers and  $\tau^2$  is the non-centrality parameter of  $Z_1$ . Then

$$(2.2) \quad \Pr\{Z_1/Z_2 \leq k\} \quad (0 < k)$$

is a monotonically decreasing function of  $\tau^2$ .

Proof. Write

$$Z_1 = Z_{11}^2 + \dots + Z_{1n_1}^2,$$

where  $Z_{1i}$ 's are independently distributed as  $N(\cdot, 1)$  with

$E Z_{11} = \tau$  and  $E Z_{1\alpha} = 0$  for  $\alpha > 1$ ; moreover  $Z_{11}$ 's are distributed

independently of  $Z_2$ . Such a decomposition of  $Z_1$  is clearly possible.

Now apply Theorem 2.1 for  $Z_{11}$  holding  $Z_2$  and  $Z_{1\alpha}$ 's for  $\alpha > 1$  fixed.

The above corollary is true also for non-integral positive  $n_1$  and  $n_2$ . One may use the monotone likelihood-ratio property of the non-central F-distribution.

Let us now consider the two special cases given by  $s = 1$  and  $p = 1$ .

Case I.  $s = 1, n_e \geq p$ . The critical region of the Hotelling's

$T^2$ -test can be expressed as

$$(2.3) \quad Y_1'(Y_{(3)}Y_{(3)})^{-1}Y_{(1)} > \{p/(n_e-p+1)\}F_{p, n_e-p+1}^\alpha$$

where  $F_{a,b}^\alpha$  is the upper  $\alpha$ -fractile of the F-distribution with  $a$

and  $b$  degrees of freedom. The power of this test is

$$(2.4) \quad \Pr[F_{p, n_e-p+1}(\tau^2) > F_{p, n_e-p+1}^\alpha],$$

where  $\tau^2 = \mu_1' \Sigma^{-1} \mu_1$ . It follows from Corollary 2.1 that the power

of this test increases monotonically with  $\tau^2$ .

Case II.  $p = 1, n_e \geq 1$ . The critical region of the ANOVA F-test

can be expressed as

$$(2.5) \quad \frac{\sum_{\alpha=1}^s Y_{\alpha}^2}{\sum_{\alpha=r+1}^n Y_{\alpha}^2} > \{s/n_e\} F_{s, n_e}^\alpha.$$

The power of this test is

$$(2.6) \quad \Pr[F_{s,n_e}(\tau^2) > F_{s,n_e}^\alpha] ,$$

where  $\tau^2 = \sum_{i=1}^s \mu_i^2 / \Sigma$ . Again, Corollary 2.1 shows that the power of this test increases monotonically with  $\tau^2$ .

### 3. Mathematical preliminaries.

The key to all the results in this paper is the following well-known inequality due to Brunn-Minkowski.

Theorem 3.1. Let  $A_1$  and  $A_2$  be two non-empty convex sets in  $R^n$ .

Then

$$(3.1) \quad V_n^{1/n}(A_1 + A_2) \geq V_n^{1/n}(A_1) + V_n^{1/n}(A_2) ,$$

where  $V_n$  stands for the  $n$ -dimensional volume, and

$$A_1 + A_2 = \{x_1 + x_2 : x_1 \in A_1, x_2 \in A_2\} .$$

This inequality was first proved by Brunn [5] in 1887 and the conditions for equality to hold were derived by Minkowski [26] in 1910. Later in 1935 Lusternik [25] generalized this result for non-empty arbitrary measurable sets  $A_1$  and  $A_2$  and derived conditions for equality to hold.

This inequality led Anderson [1] to generalize Theorem 2.1 to the multivariate case. We shall present here a minor extension of Anderson's result. Following Anderson we shall call a non-negative function  $f$  on  $R^n$  unimodal, if

$$(3.2) \quad K_{f,u} \equiv \{x \in R^n : f(x) \geq u\}$$

is convex for all  $u$ ,  $0 \leq u < \infty$ . We shall call a (real-valued) function  $f$  on  $R^n$  centrally symmetric if  $f(x) = f(-x)$  for all  $x \in R^n$ .

Theorem 3.2. Let  $G$  be a group of linear Lebesgue measure preserving transformations of  $R^n$  onto  $R^n$ . Let  $f$  be a non-negative (Borel-measurable) function on  $R^n$  such that  $f$  is unimodal, integrable with respect to the Lebesgue measure  $\mu_n$  on  $R^n$ , and  $f(x) = f(gx)$  for all  $g \in G$ ,  $x \in R^n$ . Let  $E$  be a convex set in  $R^n$  such that  $E = gE$  for all  $g$  in  $G$ . Then for any fixed  $\tau \in R^n$  and any  $\tau^*$  in the convex-hull of the  $G$ -orbit of  $\tau$  defined by  $G(\tau) \equiv \{g\tau: g \in G\}$

$$(3.3) \quad \int_{E+\tau^*} f(x)dx \geq \int_{E+\tau} f(x)dx .$$

Proof. First note that

$$(3.4) \quad \int_{E+\tau} f(x)dx = \int_0^\infty \mu_n[K_{f,u} \cap (E+\tau)]du ,$$

where  $K_{f,u}$  is defined in (3.2). Then for  $g \in G$ ,  $K_{f,u} = gK_{f,u}$ , and

$$(3.5) \quad \begin{aligned} \mu_n[K_{f,u} \cap (E+\tau)] &= \mu_n[gK_{f,u} \cap g(E+\tau)] \\ &= \mu_n[K_{f,u} \cap (E+g\tau)] . \end{aligned}$$

Note that  $K_{f,u} \cap (E+\tau)$  and  $K_{f,u} \cap (E+g\tau)$  are both either empty or non-empty. Let  $g_1, \dots, g_m$  be in  $G$  and  $\tau^* = \sum_{i=1}^m \lambda_i g_i \tau$ , where  $0 \leq \lambda_i \leq 1$ ,  $\sum_{i=1}^m \lambda_i = 1$ . Then

$$(3.6) \quad K_{f,u} \cap (E+\tau^*) \supset \sum_{i=1}^m \lambda_i [K_{f,u} \cap (E+g_i\tau)] ,$$

whenever  $K_{f,u} \cap (E+\tau)$  is non-empty. Theorem 3.1 now yields

$$(3.7) \quad \begin{aligned} \mu_n[K_{f,u} \cap (E+\tau^*)] &\geq \left[ \sum_{i=1}^m \lambda_i \mu_n^{1/n} \{K_{f,u} \cap (E+g_i\tau)\} \right]^n \\ &= \mu_n[K_{f,u} \cap (E+\tau)] . \end{aligned}$$

Integrating with respect to  $u$  yields the theorem.



We shall improve Theorem 3.2 by using a condition on  $f$  which is stronger than unimodality. Following Das Gupta [11] we shall call a non-negative function  $f$  on  $R^n$  0-unimodal (or, strongly unimodal) if for any  $x_0, x_1$  in  $R^n$  and any  $0 < \theta < 1$

$$(3.8) \quad f[(1-\theta)x_0 + \theta x_1] \geq f^{1-\theta}(x_0)f^\theta(x_1) . \quad (\text{Borel-measurable})$$

Theorem 3.3. Let  $f$  be a non-negative 0-unimodal function on  $R^n$  such that  $f$  is integrable with respect to  $\mu_n$ . Then for any two Borel-measurable non-empty sets  $E_0$  and  $E_1$

$$(3.9) \quad \int_{(1-\theta)E_0 + \theta E_1} f(x)dx \geq \left[ \int_{E_0} f(x)dx \right]^{1-\theta} \left[ \int_{E_1} f(x)dx \right]^\theta .$$

Proof. For  $u \in R^1$  define

$$(3.10) \quad C = \{(x, u) \in R^n \times R^1 : f(x) \geq \exp(-u)\} .$$

Let  $C_u$  be the  $u$ -section of  $C$ . Then for any measurable set  $E \subset R^n$

$$(3.11) \quad \int_E f(x)dx = \int_{-\infty}^{\infty} \mu_n[C_u \cap E] \exp(-u)du .$$

We assume that the integrals in the left-hand side of (3.9) are positive (excluding the trivial cases). Define

$$(3.12) \quad h_\theta(u) = \mu_n[C_u \cap \{(1-\theta)E_0 + \theta E_1\}] .$$

Let  $S_i$  be the support of  $h_i$  ( $i = 0, 1$ ). Then for  $u_0 \in S_0$ ,  $u_1 \in S_1$ ,  $u = (1-\theta)u_0 + \theta u_1$

$$(3.13) \quad h_\theta(u) \geq [h_0(u_0)]^{1-\theta} [h_1(u_1)]^\theta .$$

To see this, note that

$$(3.14) \quad C_u \cap \{(1-\theta)E_0 + \theta E_1\} \supset (1-\theta)(C_{u_0} \cap E_0) + \theta(C_{u_1} \cap E_1) .$$

From Brunn-Minkowski-Lusternik inequality we get

$$(3.15) \quad \mu_n^{1/n}[C_u \cap \{(1-\theta)E_0 + \theta E_1\}] \geq (1-\theta)\mu_n^{1/n}(C_{u_0} \cap E_0) + \theta\mu_n^{1/n}(C_{u_1} \cap E_1) .$$

Applying the arithmetic-mean geometric-mean inequality we finally get

$$(3.16) \quad \mu_n[C_u \cap \{(1-\theta)E_0 + \theta E_1\}] \geq [\mu_n(C_{u_0} \cap E_0)]^{1-\theta} [\mu_n(C_{u_1} \cap E_1)]^\theta.$$

Multiplying both the sides by

$$(3.17) \quad \exp(-u) = \exp[(1-\theta)u_0] \exp[\theta u_1]$$

we get (3.13). The following lemma will now yield the theorem.

Lemma 3.3.1. Let  $g_0$  and  $g_1$  be non-negative (Borel-measurable) integrable functions on  $R^1$  with non-empty supports given by  $S_0$  and  $S_1$ , respectively. Let  $g$  be a non-negative Borel-measurable integrable function on  $R^1$  such that for  $0 < \theta < 1$   $x = (1-\theta)x_0 + \theta x_1$ ,  $x_i \in S_i$

$$(3.18) \quad g(x) \geq g_0^{1-\theta}(x_0) g_1^\theta(x_1).$$

Then

$$(3.19) \quad \int_{(1-\theta)S_0 + \theta S_1} g(x) dx \geq \left[ \int_{S_0} g_0(x) dx \right]^{1-\theta} \left[ \int_{S_1} g_1(x) dx \right]^\theta.$$

Proof. First we shall assume that  $g_i$ 's are bounded. Let  $c_i$  be the supremum of  $g_i$ .  $C_i$ 's are assumed to be positive (excluding the trivial case). Define

$$(3.20) \quad A_i = \{x^* = (x, z) \in R^2: g_i(x) > c_i z, z > 0, x \in S_i\},$$

$i = 0, 1$ , and

$$(3.21) \quad A = \{x^* = (x, z) \in R^2: g(x) > z c_0^{1-\theta} c_1^\theta, z > 0, x \in (1-\theta)S_0 + \theta S_1\}.$$

Let  $A_i(z)$  and  $A(z)$  be the  $z$ -sections of  $A_i$  and  $A$ , respectively.

For  $0 < z < 1$  both  $A_0(z)$  and  $A_1(z)$  are non-empty, and

$$(3.22) \quad A(z) \supset (1-\theta)A_0(z) + \theta A_1(z).$$

Moreover,

$$(3.23) \quad \int_{-\infty}^{\infty} g_i(x) dx = c_i \int_0^1 \mu_1(A_i(z)) dz$$

We may assume that the integrals in the left-hand side of (3.19) are positive, the result is trivial otherwise.

$$(3.24) \quad \int_{(1-\theta)S_0 + \theta S_1} g(x) dx \geq c_0^{1-\theta} c_1^\theta \int_0^1 \mu_1(A(z)) dz .$$

By the one-dimensional Brunn-Minkowski-Lusternik inequality

$$(3.25) \quad \mu_1(A(z)) \geq (1-\theta)\mu_1(A_0(z)) + \theta \mu_1(A_1(z)) ,$$

for  $0 < z < 1$  . Now it follows that

$$(3.26) \quad \int_{(1-\theta)S_0 + \theta S_1} g(x) dx \geq c_0^{1-\theta} c_1^\theta [(1-\theta)c_0^{-1} \int_{-\infty}^{\infty} g_0(x) dx + \theta c_1^{-1} \int_{-\infty}^{\infty} g_1(x) dx] \\ \geq [\int_{-\infty}^{\infty} g_0(x) dx]^{1-\theta} [\int_{-\infty}^{\infty} g_1(x) dx]^\theta .$$

In the general case, define

$$(3.27) \quad g_{ik}(x) = \begin{cases} g_i(x), & \text{if } g_i(x) \leq k \\ k, & \text{if } g_i(x) > k \end{cases} .$$

Then  $g_{ik}(x) \uparrow g_i(x)$  as  $k \rightarrow \infty$  . Now apply the above result to  $g_{ik}$ 's and appeal to the monotone convergence theorem.

Theorem 3.4. Let  $f$  be a function on  $R^n$  satisfying the conditions in Theorem 3.3. Let  $E$  be a convex set in  $R^n$ , and for  $\tau \in R^n$  define

$$(3.28) \quad h(\tau) = \int_{E+\tau} f(x) dx$$

Then  $h$  is a 0-unimodal function on  $R^n$ , i.e.

$$(3.29) \quad h[(1-\theta)\tau_0 + \theta\tau_1] \geq h^{1-\theta}(\tau_0)h^\theta(\tau_1)$$

for  $0 < \theta < 1$ ,  $\tau_i \in R^n$  .

Proof. Apply theorem 3.3 with  $E_0 = E + \tau_0$ ,  $E_1 = E + \tau_1$ , and note that  $(1-\theta)E_0 + \theta E_1 = E + [(1-\theta)\tau_0 + \theta\tau_1]$  .

Corollary 3.4.1. Define  $h$  as in Theorem 3.4. Suppose

$$(3.30) \quad h(\tau_1) = \dots = h(\tau_m) = h(\tau)$$

for  $\tau_i$ 's and  $\tau$  in  $R^n$ . Then

$$(3.31) \quad h\left(\sum_{i=1}^m \lambda_i \tau_i\right) \geq h(\tau)$$

for  $0 \leq \lambda_i \leq 1$ ,  $\sum_{i=1}^m \lambda_i = 1$ .

#### 4. Study on monotonicity in the general case.

For studying tests in  $\Phi_G$  we shall reduce the problem further.

Recall that  $\tau_1^2, \dots, \tau_\ell^2$  are the  $\ell$  largest characteristic roots of  $\Sigma^{-1}MM'$ . It is possible to write

$$(4.1) \quad \Sigma^{-\frac{1}{2}}M = Q\Delta(\tau)L',$$

where  $Q: p \times p$  and  $L: s \times s$  are orthogonal matrices, and

$$(4.2) \quad \Delta(\tau) = \left[ \begin{array}{c|c} \Delta^*(\tau) & 0 \\ \hline 0 & 0 \end{array} \right], \quad \Delta^*(\tau) = \text{diag}(\tau_1, \dots, \tau_\ell),$$

$$\tau = (\tau_1, \dots, \tau_\ell)'.$$

Define

$$(4.3) \quad A = Q'\Sigma^{-\frac{1}{2}}, \quad U = AY_{(1)}L, \quad V = AY_{(3)}.$$

Then the columns of  $U$  and  $V$  are independently distributed as

$n_p(\cdot, I_p)$ , and  $EU = \Delta(\tau)$ ,  $EV = 0$ . Note that the nonzero characteristic roots of  $(UU')(UU'+VV')^{-1}$  are the same as those of  $S_0S_t^{-1}$ . This shows that the power function of any test in  $\Phi_G$  depends on  $\Sigma, M$  only through  $\tau$ . We shall now write  $S_0 = UU'$ ,  $S_e = VV'$ ,  $S_t = S_0 + S_e$ .

For a non-randomized test  $\varphi$ , let  $A_\varphi$  be its acceptance region.

We shall first consider acceptance regions in the space of  $U$  and  $V$ .

The power function of a test  $\varphi$  is

$$(4.4) \quad \mathcal{E}_{M,\Sigma} \varphi(U,V) = P_{M,\Sigma}[(U,V) \notin A_\varphi] .$$

For  $\varphi \in \Phi_G$  the power function of  $\varphi$  will be denoted by  $\pi(\tau; \varphi)$ . Given  $\tau_i^2$ 's and the structure of  $\Delta$  in (4.2) the diagonal elements of  $\Delta$  in (4.2) are not uniquely defined. In particular, by choosing  $Q$  and  $L$  appropriately it is possible to write in (4.2)  $\Delta = \Delta(D_e \tau)$ , as well as,  $\Delta = \Delta(\Gamma \tau)$ , where  $D_e$  is an  $\ell \times \ell$  diagonal matrix with diagonal elements as  $\pm 1$ , and  $\Gamma$  is an  $\ell \times \ell$  orthogonal permutation matrix, i.e.  $\Gamma \tau = (\tau_{i_1}, \dots, \tau_{i_\ell})'$  for some permutation  $(i_1, \dots, i_\ell)$  of  $(1, \dots, \ell)$ . Hence for  $\varphi \in \Phi_G$

$$(4.5) \quad \pi(\tau; \varphi) = \pi(D_e \tau; \varphi) = \pi(\Gamma \tau; \varphi)$$

for any such matrices  $D_e$  and  $\Gamma$  and for all  $\tau \in R^\ell$ .

Let  $U_i$  be the  $i^{\text{th}}$  column vector of  $U$  and  $\bar{U}^{(i)}$  be the matrix  $U$  with  $U_i$  deleted. For a region  $A$  in  $(U,V)$  space, let  $A(\bar{U}^{(i)}, V)$  be the section of  $A$  in the  $U_i$ -space, i.e.

$$(4.6) \quad A(u^{(i)}, v) = \{u_i \in R^p : (u, v) \in A\} .$$

For any test  $\varphi \in \Phi_G$  and all  $u^{(i)}$  and  $v$

$$(4.7) \quad A_\varphi(u^{(i)}, v) = -A_\varphi(u^{(i)}, v) ,$$

and for all  $v$

$$(4.8) \quad A_\varphi(v) = -A_\varphi(v) ,$$

where  $A_\varphi(v)$  is the section of  $A_\varphi$  in the  $u$ -space.

Later we shall require  $A_\varphi$  to be a region in the space of  $(U, VV')$ , or in the space of  $(U, UU' + VV')$ . For that purpose we denote the acceptance region of  $\varphi$  as  $\tilde{A}_\varphi$  to mean that it is a region in  $\mathbb{M}_{p,s} \times \mathbb{S}_p^+$ .

Next we shall introduce four subclasses of  $\Phi_G$  as follows:

- (1)  $\Phi_G^{(1)}$  is the set of all  $\varphi \in \Phi_G$  such that the acceptance region  $A_\varphi$  (in the space of  $U$  and  $V$ ) is convex in the space of each column vector of  $U$  for each set of fixed values of  $V$  and of the other column vectors of  $U$ , i.e. for every  $i$  and all  $\tilde{u}^{(i)}$  and  $v$  the set  $A_{\varphi}(\tilde{u}^{(i)}, v)$  is convex.
- (2)  $\Phi_G^{(2)}$  is the set of all  $\varphi \in \Phi_G$  such that the acceptance region  $A_\varphi$  is convex in the space of  $U$  for each set of fixed value of  $V$ .
- (3)  $\Phi_G^{(3)}$  is the set of all  $\varphi \in \Phi_G$  such that the acceptance region  $\tilde{A}_\varphi$  (in the space of  $(U, VV')$ ) is convex in  $U$  and  $VV'$ .
- (4)  $\Phi_G^{(4)}$  is the set of all  $\varphi \in \Phi_G$  such that the acceptance region  $\tilde{A}_\varphi$  (in the space of  $(U, S_t = UU' + VV')$ ) is convex in  $U$  and  $S_t$ . Note that  $\Phi_G^{(1)} \supset \Phi_G^{(2)} \supset \Phi_G^{(3)}$ .

Theorem 4.1. For  $\varphi \in \Phi_G^{(1)}$  the power function of  $\varphi$  given by  $\pi(\tau, \varphi)$  is a symmetric function in each  $\tau_i$  and monotonically increases as each  $|\tau_i|$  increases separately.

Proof. The first part of the theorem follows from (4.5). For  $i = 1, \dots, l$

$$(4.9) \quad \mathcal{E}_\tau[1 - \varphi(U, V) | U^{(i)} = u^{(i)}, V = v] = \int_{A_\varphi(u^{(i)}, v) + \tau_i e_o} f(u_i) du_i,$$

where  $f$  is the p.d.f. corresponding the  $n_p(0, I_p)$  and  $e_i$  is the vector in  $R^p$  with 1 at the  $i^{\text{th}}$  position and the other components being 0.

Now we shall use Theorem 3.2. Note that the density function  $f$  is unimodal and centrally symmetric.  $A_{\varphi}(u^{(i)}, v)$  is convex and centrally symmetric. Specialize  $G$  in Theorem 3.2 to be the group of sign transformations on  $R^p$ . Note that the distribution of  $U^{(i)}$  and  $V$  is free from  $\tau_i$ . Hence

$$\begin{aligned}
 & P[U_i \in A_{\varphi}(\tilde{u}^{(i)}, v) + \lambda_i \tau_i e_i | \tilde{U}^{(i)} = \tilde{u}^{(i)}, V = \tilde{v}] \\
 (4.10) \quad & = P[U_i \in A_{\varphi}(\tilde{u}^{(i)}, v) + (1+\lambda_i)\tau_i e_i/2 - (1-\lambda_i)\tau_i e_i/2 | \tilde{U}_i = \tilde{u}_i, V = v] \\
 & \geq P[U_i \in A_{\varphi}(\tilde{u}^{(i)}, v) + \tau_i e_i | \tilde{U}_i = \tilde{u}_i, V = v],
 \end{aligned}$$

where  $-1 \leq \lambda_i \leq 1$  and the conditional p.d.f. of  $U_i$  is taken as  $f$ . Taking expectation with respect to  $\tilde{U}_i$  and  $V$  we find that  $\pi(\tau; \varphi)$  increases if  $\tau_i$  is replaced by  $\lambda_i \tau_i$ , where  $-1 \leq \lambda_i \leq 1$ , holding the other components of  $\tau$  fixed.

Since  $f$  is also 0-unimodal the result would also follow from Corollary 3.4.1.

In the above theorem we need only  $n_e + s > p$ .

Corollary 4.1.1. If  $\varphi \in \mathfrak{F}_G^{(2)}$  the power function of  $\varphi$  is a symmetric function in each  $\tau_i$  and increases monotonically in each  $|\tau_i|$ .

Proof. Simply note that  $\mathfrak{F}_G^{(2)} \subset \mathfrak{F}_G^{(1)}$ .

Let  $H$  be the group of transformations acting on  $R^{\ell}$  defined as follows. For  $\tau \in R^{\ell}$ ,  $h \in H$

$$(4.11) \quad h\tau = (e_1 \tau_{i_1}, \dots, e_{\ell} \tau_{i_{\ell}}),$$

where  $e_i = \pm 1$  and  $(i_1, \dots, i_{\ell})$  is a permutation of  $(1, \dots, \ell)$ .

Theorem 4.2. If  $\varphi \in \mathfrak{F}_G^{(3)}$ , and  $\tau \in R^\ell$

$$(4.12) \quad \pi(\tau^*; \varphi) \leq \pi(\tau; \varphi),$$

where  $\tau^*$  is any point in the convex-hull of the H-orbit of  $\tau$ , provided  $n_e \geq p+1$ .

Proof. The joint density  $p_0$  of  $U$  and  $S_e = VV'$  under  $H_0$  is 0-unimodal when  $n_e \geq p+1$ . For  $h \in H$ ,  $\tau \in R^\ell$

$$(4.13) \quad \pi(h\tau; \varphi) = \pi(\tau; \varphi).$$

For  $h_i \in H$  and  $0 \leq \lambda_i \leq 1$ ,  $\sum_{i=1}^m \lambda_i = 1$

$$(4.14) \quad \sum_{i=1}^m \lambda_i \Delta(h_i \tau) = \Delta\left(\sum_{i=1}^m \lambda_i h_i \tau\right).$$

Moreover

$$(4.15) \quad \begin{aligned} P_{\tau}[(U, S_e) \in \tilde{A}_{\varphi}|H_1] \\ = P[(U + \Delta(\tau), S_e) \in \tilde{A}_{\varphi}|H_0]. \end{aligned}$$

The theorem now follows from Corollary 3.4.1.

Theorem 3.4 also yields the following.

Corollary 4.2.1. If  $\varphi \in \mathfrak{F}_G^{(3)}$  the power function of  $\varphi$  given by  $\pi(\tau; \varphi)$  is a 0-unimodal function of  $\tau$ , provided  $n_e \geq p+1$ .

Theorem 4.3. If  $\varphi \in \mathfrak{F}_G^{(4)}$  the result in Theorem 4.2 holds provided  $n_e \geq p+1$ .

Proof. The joint density of  $U$  and  $S_t$  under  $H_0$  is given by

$$(4.16) \quad \begin{aligned} q(u, s_t) &= C \exp(-\frac{1}{2}\text{tr}(s_t)) [\det(s_t - uu')]^{\frac{n_e - p - 1}{2}}, \text{ if } s_t - uu' \in S_p^+ \\ &= 0 \text{ otherwise.} \end{aligned}$$

The following facts show that  $q$  is a 0-unimodal function when  $n_e \geq p+1$



(i) If  $A_0$  and  $A_1$  are  $p \times p$  positive-definite matrices

$$(4.17) \quad \det((1-\theta)A_0 + \theta A_1) \geq (\det A_0)^{1-\theta} (\det A_1)^\theta,$$

for  $0 < \theta < 1$ .

(ii) Let  $U^{(0)}, U^{(1)}$  be elements in  $M_{p,s}$  and  $U \equiv (1-\theta)U^{(0)} + \theta U^{(1)}$  for  $0 < \theta < 1$ . Then

$$(4.18) \quad \begin{aligned} & (1-\theta)U^{(0)}U^{(0)*} + \theta U^{(1)}U^{(1)*} \\ &= UU^* + (1-\theta)\theta(U^{(0)}-U^{(1)})(U^{(0)}-U^{(1)})^*. \end{aligned}$$

(iii) If  $A_0$  and  $A_1$  are non-negative definite  $p \times p$  matrices

$$(4.19) \quad \det(A_0 + A_1) \geq \det(A_0) + \det(A_1).$$

The rest of the proof is the same as that of Theorem 4.2.

Corollary 4.3.1. If  $\varphi \in \mathfrak{F}_G^{(4)}$  the power function of  $\varphi$  given by  $\pi(\tau; \varphi)$  is a 0-unimodal function of  $\tau$ , provided  $n_e \geq p+1$ .

Next we shall study the four standard invariant tests given in Section 1.

Theorem 4.4. (a) The likelihood-ratio test is in  $\mathfrak{F}_G^{(1)}$ .

(b) Roy's maximum root test is in  $\mathfrak{F}_G^{(3)}$ .

(c) Lawley-Hotelling's trace test is in  $\mathfrak{F}_G^{(3)}$ .

(d) Pillai's trace test is in  $\mathfrak{F}_G^{(4)}$ .

(e) Pillai's trace test is in  $\mathfrak{F}_G^{(1)}$  if and only if the cut-off point  $R_4 \leq \max(1, p-n_\ell)$ .

Proof. (a) Let  $W_i = (U^{(i)}, V)$  then the acceptance region of the likelihood-ratio test can easily be expressed as

$$(4.20) \quad 1 + U_i^*(W_i W_i^*)^{-1} U_i \leq (\det V V^*)/k \det(W_i W_i^*)$$

which is clearly convex in  $U_i$  for fixed  $W_i$ .

(b) Note that

$$(4.21) \quad \begin{aligned} & \max_{a \in \mathbb{R}^p} \text{ch}[(UU')S_e^{-1}] \leq k_2 \\ & = \bigcap_{a \in \mathbb{R}^p} [(U, S_e): a'UU'a \leq k_2 a'S_e a] . \end{aligned}$$

It follows from (4.18) that the region  $a'UU'a \leq k_2 a'S_e a$  is convex in  $(U, S_e)$ .

(c) For a matrix  $B \in \mathbb{M}_{p,s}$

$$(4.22) \quad \begin{aligned} & \text{tr}(S_e^{\frac{1}{2}} B)'(S_e^{-\frac{1}{2}} U) \leq [\text{tr}(B'S_e B) \text{tr}(U'S_e^{-1} U)]^{\frac{1}{2}} \\ & \leq (\frac{1}{2}) \text{tr}(B'S_e B + U'S_e^{-1} U) . \end{aligned}$$

Hence

$$(4.23) \quad \text{tr}(B'U) - \frac{1}{2} \text{tr}(B'S_e B) \leq \frac{1}{2} \text{tr} U'S_e^{-1} U ,$$

the equality is attained when  $B = S_e^{-1} U$ . Hence the region in  $(U, S_e)$  given by  $\text{tr}(UU')S_e^{-1} \leq k_3$  is the intersection of the regions

$$(4.24) \quad \text{tr}(B'U) - \frac{1}{2} \text{tr}(B'S_e B) \leq \frac{1}{2} k_3$$

for  $B \in \mathbb{M}_{p,s}$ . However, each such region (4.24) is convex in  $(U, S_e)$ .

(d) The proof is the same as in (c).

(e) The proof of this result is rather involved and we refer to [29]. Note however that tables for  $k_h$  are partially available and even then they were obtained when  $n_e \geq p$ .

Examples of other tests in  $\Phi_G^{(i)}$  ( $i = 1, 2, 3, 4$ ) are given in [6], [27], [17], [36], [15]. A step-down test of  $H_0$  vs.  $H_1$  is given in [32]; however, this test is not in  $\Phi_G$ . This test can easily be shown to be unbiased since it is given in terms of  $F$  tests. Only partial results are known for the monotonicity property of this test; see [7] and [10].

For the case  $p = 1$  the power function of the F-test increases monotonically in  $n_e$  and decreases in  $s$  when the other parameters are held fixed. For  $s = 1$ , the power of the Hotelling's  $T^2$ -test increases if  $n_e$  increases, or if  $p$  decreases when the other parameters are held fixed. The proofs of these two results are given in [8]. Similar results for the general case are only known in very special situations; see [10], [9].

##### 5. General MANOVA models.

The general MANOVA model introduced by Potthoff and Roy [30] may be described as follows: Let  $X: p \times n$  be a random matrix such that its column vectors are independently distributed as  $N_p(\cdot, \Sigma)$  with an unknown positive-definite matrix  $\Sigma$ ; moreover  $EX' = A_1 \otimes A_2$ , where  $A_1: n \times m$  is a known matrix of rank  $r$ ,  $A_2: q \times p$  is a known matrix of rank  $q$ , and  $\otimes: m \times q$  is a matrix of unknown parameters. The problem is to test  $H_0: A_3 \otimes A_4 = 0$  against  $H_1: A_3 \otimes A_4 \neq 0$ , where  $A_3 \otimes A_4$  is bilinearly estimable, and  $A_3: s \times m$  and  $A_4: q \times v$  are known matrices of ranks  $s$  and  $v$ , respectively. This problem can be reduced to the following canonical form: Let

$$(5.1) \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{matrix} q-v \\ v \\ p-q \end{matrix}$$

$\begin{matrix} s & r-s & n-r \end{matrix}$

be a random matrix such that its column vectors are independently distributed as  $N_p(\cdot, \Sigma)$ , and

$$(5.2) \quad EY = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The problem is to test  $H_0: M_{21} = 0$  against  $M_{21} \neq 0$ .

Let us partition  $\Sigma$  as in the above.

$$(5.3) \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}$$

A class of tests invariant under a certain group of transformations which keeps the problem invariant is obtained by Gleser and Olkin [18]. However, this problem is generally viewed in the conditional set-up described below.

The column vectors of

$$(5.4) \quad \tilde{Y} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \end{bmatrix}$$

are conditionally independently distributed as  $N_q(\cdot, \tilde{\Sigma})$ , given  $Y_{31}$ ,  $Y_{32}$  and  $Y_{33}$ ;  $\tilde{\Sigma}$  is the covariance matrix of the first  $q$  components given the last  $p-q$  components derived from  $\Sigma$ . The conditional expectation of  $\tilde{Y}$  is

$$(5.5) \quad E^*(\tilde{Y}) = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \end{bmatrix} + \beta [Y_{31} \ Y_{32} \ Y_{33}] ,$$

where  $\beta$  is the matrix of regression coefficients. In this conditional set-up the s.p. matrices due to error and the hypothesis  $H_0$  are respectively defined by (assuming  $n-r \geq p-q$ )

$$(5.6) \quad S_e = Y_{23} Y'_{23} - Y_{23} Y'_{33} (Y_{33} Y'_{33})^{-1} Y_{33} Y'_{23}$$

$$(5.7) \quad S_0 = \hat{M}_{21} (I_s + Y'_{31} (Y_{33} Y'_{33})^{-1} Y_{31})^{-1} \hat{M}_{21}' ,$$

where

$$(5.8) \quad \hat{M}_{21} = Y_{21} - Y_{23} Y'_{33} (Y_{33} Y'_{33})^{-1} Y_{31} .$$

In the conditional situations  $S_e$  and  $S_0$  are independently distributed as the Wishart distributions  $W_v(n-r-p+q, \Sigma_{22 \cdot 3})$  and  $W_v(s, \Sigma_{22 \cdot 3}; \tilde{\Delta})$ , respectively, where  $\Sigma_{22 \cdot 3}$  is the covariance matrix of the second set (of  $v$ ) components given the third set of  $(p-q)$  components, and

$$(5.9) \quad \tilde{\Delta} = M_{21}(I_s + Y'_{31}(Y_{33}Y'_{33})^{-1}Y_{31})^{-1}M'_{21}.$$

As in the MANOVA one might consider those tests which depend only on the characteristic roots of  $S_0 S_e^{-1}$ . In particular, the acceptance region of the likelihood-ratio test is given by  $|S_e|/|S_0 + S_e| \geq k$ .

The column vectors of  $(Y_{31} \ Y_{33})$  are independently distributed as  $N_{p-q}(0, \Sigma_{33})$ . It is clear that the distribution of  $Y'_{31}(Y_{33}Y'_{33})^{-1}Y_{31}$  does not depend on  $\Sigma_{33}$  and we shall assume it to be  $I_{p-q}$ . Also for considering the distribution of the roots of  $S_0 S_e^{-1}$  we might take  $\Sigma_{22 \cdot 3} = I_v$  and replace  $M_{21}$  by  $\Sigma_{22 \cdot 3}^{-\frac{1}{2}} M_{21}$ . As in the MANOVA case, we can replace  $\Sigma_{22 \cdot 3}^{-\frac{1}{2}} M_{21}$  by a matrix  $\Delta: v \times s$  such that

$$(5.10) \quad \Delta = \left[ \begin{array}{c|c} \text{diag}(\tau_1, \dots, \tau_\ell) & 0 \\ \hline 0 & 0 \end{array} \right]$$

where  $\ell = \min(v, s)$  and  $\tau_i^2$  ( $\tau_i > 0$ ) are the characteristic roots of  $M'_{21} \Sigma_{22 \cdot 3}^{-1} M_{21}$ . This discussion leads us to take  $\tilde{\Delta}$  as

$$(5.11) \quad \tilde{\Delta} = \Delta(I_s + Y'_{31}(Y_{33}Y'_{33})^{-1}Y_{31})^{-1}\Delta'.$$

Arguing as in Anderson and Das Gupta [3] we see that the characteristic roots of  $\tilde{\Delta}$  increase if any  $\tau_i$  is increased. Thus all the results in the MANOVA case can be applied now.

## 6. Bibliographical Notes.

On Section 1. For a general discussion of MANOVA see Anderson [2], Roy [33], and Lehmann [23].

On Section 2. See Roy [33].

On Section 3. A proof of Theorem 3.1 is given in Bonneson and Fenchel [4]. For Lusternik's generalization of Theorem 3.1 see Hadwiger and Ohman [19] or Henstock and Macbeath [20].

Theorem 3.2 was proved by Anderson [1] when  $G$  is the group of sign transformations. Essentially the same proof also holds for any  $G$  defined in Theorem 3.2; the general statement is due to Mudholkar [28]. For further generalizations of this theorem see Das Gupta [11].

Theorem 3.3 was proved by Prekopa [31] and Leindler [24] (for  $n = 1$ ); however, their proofs are quite obscure and somewhat incomplete. The present proof uses essentially the ideas given by Henstock and Macbeath [20]; see Das Gupta [13] for more general results. Theorem 3.4 was proved by Ibragimov [21] and Schoenberg [35] when  $n = 1$ ; the general case was proved by Davidovic, Korenbljum and Hacet [14]. For a discussion of these results see Das Gupta [13].

On Section 4. Theorem 4.1 is due to Das Gupta, Anderson and Mudholkar [6] where the monotonicity property of the power functions of tests (a), (b), and (c) are established. Roy and Mikhail [34] also proved the monotonicity property of the maximum root test. (Srivastava [37]) derived the result for tests (a)-(c) although his proofs are incomplete. Theorem 4.2 and its present proof are due to Das Gupta [12]; an alternative proof using Theorem 3.2 is given by Eaton and Perlman [15].

On Section 5. See Fujikoshi [16] and Khatri [22].

References

- [1] Anderson, T. W. (1955) The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Amer. Math. Soc. 6, 170-176.
- [2] Anderson, T. W. (1958) An Introduction to Multivariate Statistical Analysis. Wiley, New York.
- [3] Anderson, T. W. and Das Gupta, S. (1964) Monotonicity of the power functions of some tests of independence between two sets of variates. Ann. Math. Statist. 35, 206-208.
- [4] Bonneson, T. and Fenchel, W. (1948) Konvexe Körper. Chelsea, New York.
- [5] Brunn, H. (1887) Über Ovale und Eiflachen, Inaugural dissertation, München.
- [6] Das Gupta, S., Anderson, T. W., and Mudholkar, G. S. (1964) Monotonicity of the power functions of some tests of the multivariate linear hypothesis. Ann. Math. Statist., 35, 200-205.
- [7] Das Gupta, S. (1970) Step-down multiple-decision rules. Essays in Probability and Statistics., Univ. of North Carolina Press, Chapel Hill.
- [8] Das Gupta, S. (1972) Noncentral matrix-variate beta distribution and Wilks' U-distribution. Sankhyā, Series A, 34, 357-362.
- [9] Das Gupta, S. and Perlman, M. D. (1973) On the power of Wilks' U-test for MANOVA. Jour. Multiv. Anal., 3, 220-225.
- [10] Das Gupta, S. and Perlman, M. D. (1974) Power of the noncentral F-test: Effect of additional variates on Hotelling's  $T^2$ -test Jour. Amer. Statist. Assoc., 69, 174-180.
- [11] Das Gupta, S. (1976) A generalization of Anderson's theorem on unimodal functions. Proc. Amer. Math. Soc., 60, 85-91.

- [12] Das Gupta, S. (1977) s-unimodal functions: related inequalities and statistical applications. Sankhyā, Series B.
- [13] Das Gupta, S. (1978) Brunn-Minkowski inequality and its aftermath. Tech. Report #310, School of Statistics, Univ. of Minnesota.
- [14] Davidovic, Ju. S., Korenbljum, B. I. and Hacet, B. I. (1962) A property of logarithmically concave functions. Soviet. Math. Dokl., 10, #2, 477-480.
- [15] Eaton, M. and Perlman, M. D. (1974) A monotonicity property of the power functions of some invariant tests. Ann. Statist., 2, 1022-1028.
- [16] Fujikoshi, Y. (1973) Monotonicity of the power functions of some tests in general MANOVA models. Ann. Statist., 1, 388-391.
- [17] Ghosh, M. N. (1964) On the admissibility of some tests of MANOVA. Ann. Math. Statist., 35, 789-794.
- [18] Gleser, L. and Olkin, I. (1970). Linear model in multivariate analysis. Essays in Probability and Statistics. Univ. of North Carolina Press, Chapel Hill.
- [19] Hadwiger, H. and Ohman, D. (1956) Brunn-Minkowskischer Satz und Isoperimetrie. Math. Zeit., 66, 1-8.
- [20] Henstock, R. and Macbeath, A. M. (1953) On the measure of sum sets (I). The theorem of Brunn, Minkowski and Lusternik. Proc. Lond. Math. Soc., 3, 182-194.
- [21] Ibragimov, I. A. (1956) On the composition of unimodal distributions. Theor. Prob. Appl. (Translation) 1, 255-266.
- [22] Khatri, C. G. (1966) A note on MANOVA model applied to problems in growth curves. Ann. Inst. Math. Statist., 18, 75-86.
- [23] Lehmann, E. L. (1959) Testing Statistical Hypotheses, Wiley, New York.
- [24] Leindler, L. (1972) On a certain converse of Holder's inequality II. Acta Scient. Mat., 33, 217-223.



- [25] Lusternik, L. (1925) Die Brunn-Minkowskische Ungleichung fur  
Beliebige Nessabare Mengen. Comptes Rendus (Doklady) de  
l'Academie des Sciences de l'URSS, Vol. III, (viii) 55-58.
- [26] Minkowski, H. (1910) Geometrie der Zahlen, Keipzig and Berlin.
- [27] Mudholkar, G. S. (1965) A class of tests with monotone power  
functions for two problems in multivariate statistical analysis  
Ann. Math. Statist., 36, 1794-1801.
- [28] Mudholkar, G. S. (1966) The integral of an invariant unimodal  
function over an invariant convex set--an inequality and  
applications. Proc. Amer. Math. Soc., 17, 1327-1333.
- [29] Perlman, M. D. (1974) Monotonicity of the power function of Pillai's  
trace test. Jour. Multiv. Anal., 4, 22-30.
- [30] Potthoff, R. F. and Roy, S. N. (1964) A generalized multivariate  
analysis of variance model useful especially for growth  
curve problems. Biometrika, 51, 313-326.
- [31] Prekopa, A. (1973) On logarithmic concave measures and functions.  
Acta Scient. Mat., 34, 335-343.
- [32] Roy, J. (1958) Step-down procedure in multivariate analysis.  
Ann. Math. Statist., 29, 1177-1187.
- [33] Roy, S. N. (1957). Some aspects of Multivariate Analysis,  
Wiley, New York.
- [34] Roy, S. N. and Mikhail, W. F. (1961) On the monotonic character  
of the power functions of two multivariate tests. Ann.  
Math. Statist., 32, 1145-1151.
- [35] Schoenberg, I. J. (1951). On Polya frequency functions I: the  
totally positive functions and their Laplace transforms.

- [36] Schwartz, R. E. (1967) Admissible tests in multivariate analysis of variance, Ann. Math. Statist., 38, 698-710.
- [37] Srivastava, J. N. (1964) On the monotonicity property of the three multivariate tests for Multivariate Analysis of Variance. J. Roy. Statist. Ser. B., 26, 77-81.